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ON LOCALLY DIVIDED RINGS AND GOING-DOWN RINGS

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1 INTRODUCTION

All rings considered below are commutative with 1. As in [B2], if R is a ring, $P \in \mathrm{Spec}(R)$ is a *divided prime ideal in* R if P is comparable under inclusion to each ideal of R; and R is a *divided ring* if each $P \in \mathrm{Spec}(R)$ is divided in R. Divided rings generalize the divided domains introduced in [D2]. Our main goal is to generalize another class of domains introduced in [D2], the locally divided domains. We say that a ring R is a *locally divided ring* if R_P is a divided ring for each $P \in \mathrm{Spec}(R)$. Each divided ring is locally divided [B2, Proposition 4]. Since the literature on locally divided domains ([D2], [D4], [DF]) is tied to studies of going-down domains (in the sense of [D1]), it is natural to pursue connections between locally divided rings and the recently introduced going-down rings [D5]. Section 3 develops several such connections, some with the flavor of domain-theoretic studies and others differing from such phenomena in the presence of zero-divisors. First, Section 2 develops more information about divided rings and initiates the theory of locally divided rings.

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A useful family of locally divided rings is given by the rings of (Krull) dimension 0 (Corollary 2.2). In contrast to the situation for domains, neither one-dimensional rings nor Prüfer rings (in the sense of [G]) need be locally divided (Example 2.18). More domain-like behavior is present if we assume a ring R satisfies $Z(R) = \mathrm{Nil}(R)$, a hypothesis which has been useful in [D5, Corollary 2.6]. (As usual, if R is a ring, then $Z(R)$ denotes the set of zero-divisors of R and $\mathrm{Nil}(R)$ denotes the nilradical of R. As noted in [D5, Proposition 2.3(a)], $Z(R) = \mathrm{Nil}(R)$ if and only if 0 is a primary ideal of R.) Proposition 2.19(b) establishes that any Prüfer ring R satisfying $Z(R) = \mathrm{Nil}(R)$ must be locally divided. Moreover, Theorem 2.7 generalizes [D6, Proposition 2.12] by characterizing, given $Z(R) = \mathrm{Nil}(R)$, when a CPI-extension (in the sense of [BS]) $R + PR_P$ is locally divided. (As explained by Proposition 2.5(c), the "$Z(R) = \mathrm{Nil}(R)$" hypothesis allows us to study locally divided rings via pullback-theoretic methods, such as [F, Theorem 1.4]). Another noteworthy result in the presence of the "$Z(R) = \mathrm{Nil}(R)$" hypothesis is Theorem 2.10, generalizing the characterizations of locally divided domains [D4, Theorem 2.4], including flatness of CPI-overrings.

Most of our non-domain examples of locally divided rings are provided by the idealization construction $R(+)E$ arising from an R-module E. We assume familiarity with idealization, as in [H, Theorem 25.1, Corollary 25.5(2)]. In Proposition 2.16, idealization is used to characterize locally divided rings. The upshot for domains R is in Corollary 2.17(b): R is a locally divided domain if and only if $R(+)E$ is a locally divided ring for each (some) vector space E over the quotient field of R.

For the work leading up to corollary 2.17, when R is a quasilocal ring, the role of "quotient field of R" is played by $R_{\mathrm{Nil}(R)}$. This localization is meaningful for the rings of interest, as they are divided, and hence, treed. (Generalizing domain-theoretic usage [D1], we say a ring R is a *treed ring* if $\mathrm{Spec}(R)$, as a poset under inclusion, is a tree; that is, if P and Q are comparable under inclusion whenever P and Q are prime ideals of R which are contained in a common maximal ideal of R). The point is that $\mathrm{Nil}(R) \in \mathrm{Spec}(R)$ whenever R is a quasilocal treed ring. A reference on quasilocal treed rings is [B1]. Since any divided ring is quasilocal and treed [B2, Proposition 1(a)], so is any pseudo-valuation ring. (PVRs were introduced in [BAD] as a generalization of the PVDs in [HH]. More generally, ϕ-PVRs were introduced in [B3]; and ϕ-PVR must be divided [B3, Proposition 4] but need not be a PVR [B4, Theorem 2.6].) As described in Remark 2.4(b), [ABD, Example 3.16(c)] shows how dramatically "$Z(R) = \mathrm{Nil}(R)$" condition can fail in a PVR. This explains why we occasionally assume that some prime ideals being considered are regular ideals.

We devote part of Section 2 to studying overrings without the "$Z(R) = \mathrm{Nil}(R)$" hypothesis. Attention is paid to the large and regular quotient rings, $R_{[P]}$ and $R_{(P)}$, of Griffin [G]. Corollary 2.24(c) establishes that if R is a ring, then $R_{[P]}$ is a divided ring for each $P \in \mathrm{Spec}(R)$ if and only if $R_{(P)}$ is a divided ring for each $P \in \mathrm{Spec}(R)$. Any such R must be locally divided.

We turn to a summary of Section 3. The theory of going-down rings is complicated by the fact that, in contrast to the situation for domains, a going-down ring need not be treed [D5, Example 1]. However, Proposition 3.1 establishes that each locally divided ring is a treed going-down ring. Corollary 3.5 obtains the converse if R is seminormal (in the sense of [S]). More generally, Theorem 3.4 gives a result that is new even for domains: a reduced ring R is a treed going-down ring if and only if the seminormalization of R (in the sense of [S]) is locally divided. Also, given the "$Z(R) = \mathrm{Nil}(R)$" hypothesis, Theorem 3.3 generalizes the characterization of quasilocal going-down domains [D2, Theorem 2.5] in terms of divided integral unibranched overrings. The proof of Theorem 3.3 uses Theorem 2.7, the pullback-theoretic result on locally divided CPI-extensions. New examples of going-down rings are available because of the characterization in Propostion 3.8 of going-down rings in terms of idealization. The most striking consequence, in contrast to the situation for domains (M2, Corollary 11], [D2, Corollary 2.8]) is Example 3.10, giving a family of quasilocal integrally closed treed going-down rings which fail to be (locally) divided.

We assume that all ring extensions and modules are unital. For a ring A, we let $\mathrm{Spec}(A)$ denote the set of prime ideals of A; $\mathrm{Max}(A)$ the set of maximal ideals of A, $\mathrm{Min}(A)$ the set of minimal prime ideals of A; $\dim(A)$ the Krull dimension of A; and $\mathrm{tq}(A)$ the total quotient ring of A. By an *overring* of a ring A, we mean any ring B such that $A \subseteq B \subseteq \mathrm{tq}(A)$. If E is an R-module, then $Z_R(E) := \{e \in E : \text{there exists } r \in R \backslash \{0\} \text{ such that } re = 0\}$.

2 LOCALLY DIVIDED RINGS

It is known [B2, Proposition 4, Corollary 3] (cf. also [D5, Remark(c), page 47]) that the class of divided rings is stable under the formation of rings of fractions and homomorphic images. Proposition 2.1 extends several similar facts about divided rings to the context of locally divided rings.

Proposition 2.1. (a) *The class of locally divided rings is stable under the formation of rings of fractions and homomorphic images.*

(b) *Let $R_1, \ldots, R_n$ be finitely many rings, and put $A = R_1 \times \cdots \times R_n$. Then A is a locally divided ring if and only if R_i is a locally divided ring for each $i = 1, \ldots, n$.*

(c) *A ring R is locally divided if and only if R_M is (locally) divided for each $M \in \mathrm{Max}(R)$.*

(d) *Each locally divided ring is a treed ring.*

Proof. (a) Let R be a locally divided ring. If S is a multiplicatively closed subset of R and $Q \in \mathrm{Spec}(R_S)$, write $Q = PR_S$, with $P \in \mathrm{Spec}(R)$ such that $P \cap S = \phi$. By [Bo, Proposition 11 (iii), page 70], $(R_S)_Q \cong R_P$ and, hence, is divided. Thus, R_S is locally divided.

It remains to show that if I is an ideal of (the locally divided ring) R and $P \in \mathrm{Spec}(R)$ contains I, then $B = (R/I)_{P/I}$ is a divided ring. As [Bo, Proposition 11(i), page 70] provides a ring-isomorphism $R_P/IR_P \to B$, the assertion follows from the above remarks.

(b) The "only if" assertion follows by applying the second assertion in (a) to the canonical projection map $R \to R_i$. For the "if" assertion, we may take $n = 2$. As $\mathrm{Spec}(A) \cong \mathrm{Spec}(R_1) \coprod \mathrm{Spec}(R_2)$, observe that if $P \in \mathrm{Spec}(R_1)$ and $Q = P \times R_2 \in \mathrm{Spec}(A)$, then $A_Q \cong (R_1)_P$.

(c) If $P \in \mathrm{Spec}(R)$, pick $M \in \mathrm{Max}(R)$ such that $P \subseteq M$, and note that $R_P \cong (PR_M)^{-1}(R_M)$ is a ring of fractions of a divided ring.

(d) Let R be locally divided. It suffices to prove that if $P, Q \in \mathrm{Spec}(R)$ and $M \in \mathrm{Max}(R)$ contains both P and Q, then P and Q are comparable under inclusion. As R_M is divided, its prime ideals are linearly ordered by inclusion [B2, Proposition 1(a)]. If $PR_M \subseteq QR_M$ and $\tau : R \to R_M$ is the canonical map, then $P = \tau^{-1}(PR_M) \subseteq \tau^{-1}(QR_M) = Q$. $\square$

Corollary 2.2. *Each zero-dimensional ring is locally divided.*

Proof. Any zero-dimensional quasilocal ring is divided. $\square$

We make three remarks about the preceding material. First, the ideas underlying the proof of Proposition 2.1(d) show that a ring R is treed if and only if R_P is treed for each prime (resp., maximal) ideal P of R. Second, the converse of Proposition 2.1(d) is false, even for domains; Example 3.10 presents a counterexample to that converse with a non-domain flavor. Third, in contrast to the case of domains, Corollary 2.2 cannot be extended to one-dimensional rings (Example 2.18 (b)).

Proposition 2.3 generalizes [B2, Lemma 8].

Proposition 2.3. *Let R be a ring and P a regular prime ideal of R. If P is comparable under inclusion to each prime ideal of R, then $Z(R) \subseteq P$.*

Proof. Since $R \backslash Z(R)$ is saturated, $Z(R) = \cup P_i$ for some set $\{P_i\} \subseteq \mathrm{Spec}(R)$. Since P is regular, $P \nsubseteq P_i$ for each i. Hence, $P_i \nsubseteq P$ for each i, and so $Z(R) = \subseteq P$. $\square$

Remark 2.4. (a) Let R be a quasilocal treed ring (for instance, a divided ring). Then $Z(R) \in \mathrm{Spec}(R)$ and $\mathrm{Min}(R) = \{\mathrm{Nil}(R)\}$ (cf. [k, Theorem 9]).

(b) Consider non-negative integers $i \leq n$. Then [ABD, Example 3.16(c)] constructs a pseudo-valuation (divided) ring R such that $\dim(R) = n$ and the height of $Z(R)$ is i. (By (a), $Z(R)$ is a prime ideal of R, as is $\mathrm{Nil}(R)$). If $i > 0$, then $P = \mathrm{Nil}(R) \subset Z(R)$, and so we cannot delete the "regular" hypothesis in Proposition 2.3.

Recall from [G] that if R is a ring and $P \in \mathrm{Spec}(R)$, then the *large (resp., regular) quotient ring of R with respect to P* is $R_{[P]} = \{u \in tq(R)$: there exists $s \in R\backslash P$ with $su \in R\}$; resp., $R_{(P)} = R_S$, where $S = (R\backslash P) \cap (R\backslash Z(R))$. In general, $R \subseteq R_{(P)} \subseteq R_{[P]} \subseteq tq(R)$.

Proposition 2.5. *Let R be a ring. Then:*

(a) *Let $P \in \mathrm{Spec}(R)$ such that $Z(R) \subseteq P$. Then the canonical R-algebra homomorphisms $R \to R_P$ and $R_P \to R_{R\backslash Z(R)} = tq(R)$ are injections, and thus permit R_P to be identified with an overring of R. Under this identification, $R_P = R_{[P]} = R_{(P)}$.*

(b) *Let $P \subseteq Q$ be prime ideals of R such that $Z(R) \subseteq P$. As in* (a), *identify R_P and R_Q with overrings of R. Then $R_Q \subseteq R_P$.*

(c) *Let $P \in \mathrm{Spec}(R)$ such that $Z(R) \subseteq P$. Identify R_P with an overring of R, as in* (a). *Then P is a divided prime ideal in R if and only if $PR_P = P$.*

(d) [*D5, Proposition* 2.3(b)] *Suppose $Z(R) = \mathrm{Nil}(R)$. If P dentoes this unique minimal prime of R, identify R_P with an overring of R, as in* (a). *Then $R_P = tq(R)$.*

Theorem 2.7 generalizes a result [*D6, Proposition* 2.12] on CPI-extension overrings of domains.

Lemma 2.6. *Let R be a ring such that $Z(R) = \mathrm{Nil}(R)$. Then each overring T of R satisfies $Z(T) = \mathrm{Nil}(T)$.*

Proof. If $u \in Z(T)$, then $u = r/s$, for some $r \in Z(R)$, $s \in R\backslash Z(R)$. As r is nilpotent, so is u. □

Theorem 2.7. *Let R be a ring such that $Z(R) = \mathrm{Nil}(R)$ and let $P \in \mathrm{Spec}(R)$. Then the following conditions are equivalent*:

(1) *Both R/P and R_P are divided (resp., locally divided) rings*;

(2) *$R + PR_P$ is divided (resp., locally divided).*

Proof. Viewing R_P as an overring of R as in Proposition 2.5(a), we see that $A = R + PR_P$ is the overring of R given by the pullback $R_P \times_k R/P$ arising from the canonical surjection $R_P \to k = R_P/PR_P$ and the canonical inclusion

$R/P \to k = tq(R/P)$. Put $Q = PR_P$. Localizing the vertices in the pullback description $A = R_P \times_k R/P$ at the multiplicative sets generated by $A \backslash Q$ leads, by [F, Proposition 1.9], to $A_Q = R_P \times_k tq(R/P)$. Thus, we have $A_Q \cong R_P$ canonically. (Cf. also [BS, Proposition 2.5, Theorem 2.4].) Identifying via this isomorphism, we conclude that $QA_Q = QR_P = Q$. Since Lemma 2.6 ensures that $Z(A) = \mathrm{Nil}(A)$, Proposition 2.5(c) yields that Q is a divided prime of A. Moreover, $A/Q \cong R/P$. Thus, we may replace R and P with A and Q, respectively, i.e., we may assume P is a divided prime of R.

We adapt the proof of [D6, Proposition 2.12]. By [B2, Corollary 3, Proposition 4] (resp., Proposition 2.1(a)), (2) $\Rightarrow$ (1). Conversely, assume (1). As in [D6, page 324, lines 6–12], we see that the "locally divided" assertion follows from the "divided" assertion. Thus, without loss of generality, R/P and R_P are divided rings and, by Proposition 2.5(c), we must show $QR_Q = Q$ for each $Q \in \mathrm{Spec}(R)$. There are two cases, according as to whether $Q \subseteq P$ or $P \subseteq Q$. The proofs for these cases, as given in [D6, page 324], carry over *verbatim* (change "domain" to "ring"), provided that their calculations are interpreted with the aid of Proposition 2.5(b) (which applies as needed, thanks to Lemma 2.6). □

Corollary 2.8. *Let R be a locally divided ring such that $Z(R) = \mathrm{Nil}(R)$. Then $R + PR_P$ is a locally divided ring for each $P \in \mathrm{Spec}(R)$.*

Proof. Combine Theorem 2.7 and Proposition 2.1(a). □

Theorem 2.10 extends [D4, Theorem 2.4], which gave characterizations of locally divided domains, to rings R satisfying $Z(R) = \mathrm{Nil}(R)$. Adapting the approach in [D4], we first extend [D4, Proposition 2.3].

Proposition 2.9. *Let R be a ring such that $Z(R) = \mathrm{Nil}(R)$ and let $P \in \mathrm{Spec}(R)$. Then $T = R + PR_P$ is a flat R-module if and only if T is a ring of fractions of R.*

Proof. We only address the "only if" assertion. Pullback-theoretic considerations lead to $\mathrm{Max}(T) = \{N \in \mathrm{Spec}(T) : P \subseteq N \cap R$ and $N \cap R \in \mathrm{Max}(R)\}$. By globalization, $T = \cap\{T_N : N \in \mathrm{Max}(T)\}$. (This intersection is interpreted via Proposition 2.5(a), which applies thanks to Lemma 2.6 and also ensures that $T_N = T_{[N]}$ for each $N \in \mathrm{Max}(T)$.) If T is R-flat, $N \in \mathrm{Max}(T)$ and $M = N \cap R$, a characterization of flat overrings [G, Proposition 10] gives $T_{[N]} = R_{[M]}$ and so, by Proposition 2.5(a), $T_N = R_M$. If $M \in \mathrm{Max}(R)$ and $P \subseteq M$, some $N \in \mathrm{Max}(T)$ satisfies $N \cap R = M$ (reason via pullbacks). The upshot is that $T = \cap\{R_M : M \in \mathrm{Max}(R),\ P \subseteq M\}$. The rest of the proof of [D4, Proposition 2.3] carries over *verbatim* (change "domain" to "ring"). □

Theorem 2.10. *Let R be a ring such that $Z(R) = \mathrm{Nil}(R)$. Then the following conditions are equivalent*:

(1) *$R + PR_P$ is R-flat for each $P \in \mathrm{Spec}(R)$*;
(2) *$R + PR_P$ is a ring of fractions of R for each $P \in \mathrm{Spec}(R)$*;
(3) *$R + PR_P \subseteq R + QR_Q$ for all comparable prime ideals $P \subseteq Q$ of R*;
(4) *$PR_P \subseteq QR_Q$ for all comparable primes $P \subseteq Q$ of R*;
(5) *$PR_P \subseteq R_Q$ for all comparable primes $P \subseteq Q$ of R*;
(6) *R is a locally divided ring.*

Proof. The ideals, rings and inclusions in (1)–(6) are interpreted via Proposition 2.5(a). Now, (1) $\Longleftrightarrow$ (2) by Proposition 2.9. With Proposition 2.5 in mind, one need only augment the proof of [D4, Theorem 2.4] with the following observations. To prove (6) $\Rightarrow$ (1): if $P \subseteq M$ and $M \in \mathrm{Max}(R)$, then $T_{R\backslash M} = R_M + PR_P$. To prove (5) $\Rightarrow$ (4), note that if $P \subseteq Q$ are primes of R and $p/r = a/s \in tq(R)$ with $p \in P$, $r \in R \backslash P$, $a \in P$ and $s \in R \backslash Q$, then $a \in P$. □

Remark 2.4(b) noted that "$Z(R) = \mathrm{Nil}(R)$" may fail for a divided ring. It is thus of interest that $\mathrm{Nil}(R)$ plays a role in characterizing (locally) divided rings.

Proposition 2.11. *Let R be a ring. Then the following conditions are equivalent*:

(1) *$\mathrm{Nil}(R)$ is a divided prime ideal of R and R/P is a divided domain for each $P \in \mathrm{Spec}(R)$*;

(2) *$\mathrm{Nil}(R)$ is a divided prime ideal of R and $R/\mathrm{Nil}(R)$ is a divided domain*;

(3) *There is an ideal I of R such that $I \subseteq \mathrm{Nil}(R)$, I is comparable under inclusion to each (resp., each principal) ideal of R and R/I is a divided ring*;

(4) *R is a divided ring.*

Proof. (4) $\Rightarrow$ (1): Combine Remark 2.4(a) and [B2, Corollary 3].

(1) $\Rightarrow$ (2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (4): We use the following criterion [B2, Proposition 2]: a ring A is divided if and only if for all $(a, b) \in A \times A$, either $a|b$ or $b|a^n$ for some $n \geq 1$. Assume (3), and fix $(a, b) \in R \times R$. Without loss of generality, $a \notin I$, since $b|0$. By (3), $I \subset Ra$, and so we may assume that $b \notin I$ (*lest* $a|b$). Put $B = R/I$, and consider $\alpha = a + I$, $\beta = b + I \in B$. Since B is divided, either $\alpha|\beta$ or $\beta|\alpha^n$ for some $n \geq 1$. Suppose first that $\alpha|\beta$. Then $b = ac + w$ for some $c \in R$, $w \in I$. As $I \subseteq Ra$, we have $w = dc$ for some $d \in R$. Then $b = a(c + d)$; that is, $a|b$. Next, if $\beta|\alpha^n$ a similar argument, using $I \subseteq Rb$, yields $b|a^n$. □

Remark 2.12. (a) The hypothesis that $\mathrm{Nil}(R)$ is divided cannot be deleted from the conditions in Proposition 2.11. To see this, let D be a divided domain which is not a field, and consider the idealization $R = D\,(+)\,D$. Observe that $\mathrm{Nil}(R) = 0\,(+)\,D \in \mathrm{Spec}(R)$ since $R/\mathrm{Nil}(\mathrm{R}) \cong \mathrm{D}$. Moreover, each $P \in \mathrm{Spec}(R)$ takes the form $P = I\,(+)\,D$ for some $I \in \mathrm{Spec}(D)$, [H, Theorem 25.1(3)], and $R/P \cong D/I$ is a divided domain [D2, Lemma 2.2(c)]. Thus, R satisfies conditions (1), (2) and (3) in Proposition 2.11. However, R does not satisfy condition (4): specifically, $\mathrm{Nil}(R)$ is not divided in R. Indeed, if we choose a nonzero nonunit $d \in D$, then $\delta = (d, 0) \notin \mathrm{Nil}(R)$ and, since $R\delta = \{(d_1 d, d_2 d) : d_1, d_2 \in D\}$, $(0, 1) \in \mathrm{Nil}(R)\backslash R\delta$.

(b) Proposition 2.11 globalizes to yield the following result. A ring R is locally divided if and only if $\mathrm{Nil}(R)R_M$ is comparable under inclusion to each (resp., each principal) ideal of R_M for each $M \in \mathrm{Max}(R)$ and $R/\mathrm{Nil}(R)$ is a locally divided ring.

We next study (locally) divided idealizations.

Lemma 2.13. *Let R be a ring and E an R-module. Suppose that $A = R\,(+)\,E$ is a divided ring. Then:*

(a) *R is a divided ring.*

(b) *Suppose that $r \in R\backslash\mathrm{Nil}(R)$, $e \in E$, $re = 0$ implies $e = 0$. Then, the R-module structure on E is induced by an $R_{\mathrm{Nil}(R)}$-module structure on E.*

(c) *Suppose that $Z(R) = \mathrm{Nil}(R)$ and E is a torsion-free R-module. Then the R-module structure on E is induced by a $tq(R)$-module structure on E.*

Proof. Since $A/(0\,(+)\,E) \cong R$, (a) follows from the fact that any homomorphic image of a divided ring is divided [B2, Corollary 3]. Also, (c) follows from (b), for $Z(R) = \mathrm{Nil}(R)$ implies that $R_{\mathrm{Nil}(R)} = tq(R)$.

(b) Consider the ring homomorphism $g : R \to \mathrm{Hom}_2(E, E)$ embodying the R-module structure of E. By (a) and Remark 2.4(a), $\mathrm{Nil}(R) \in \mathrm{Spec}(R)$. It suffices to show that if $r \in R\backslash\mathrm{Nil}(R)$, then $g(r)$ is a bijection. The hypothesis in (b) ensures that $g(r)$ is an injection. It therefore suffices to prove that if $e \in E$, then there exists $f \in E$ such that $rf = e$.

Observe that $Q = \mathrm{Nil}(A) = \mathrm{Nil}(R)(+)E \in \mathrm{Spec}(A)$ since $A/Q \cong R/\mathrm{Nil}(R)$. Hence, Q is divided in A. As $a = (r, 0) \in A\backslash Q$, we have $Q \subseteq Aa$. Thus, there exists $b = (s, f) \in A$ such that $(0, e) = ba = (sr, rf)$, whence $e = rf$. $\square$

Proposition 2.14. *Let R be a ring. Then the following conditions are equivalent:*

(1) $\mathrm{Nil}(R) \in \mathrm{Spec}(R)$ *and $R\,(+)\,E$ is a divided ring for each $R_{\mathrm{Nil}(R)}$-module E;*

(2) $\mathrm{Nil}(R) \in \mathrm{Spec}(R)$ *and* $R\,(+)\,E$ *is a divided ring for some* $R_{\mathrm{Nil}(R)}$-*module* E;

(3) R *is a divided ring*.

Proof. "$\mathrm{Nil}(R) \in \mathrm{Spec}(R)$" is included in conditions (1) and (2) in order that the localization $R_{\mathrm{Nil}(R)}$ be meaningful. Now, (1) $\Rightarrow$ (2) trivially; and (2) $\Rightarrow$ (3) by Lemma 2.13(a). It remains to prove that (3) $\Rightarrow$ (1). Assume (3). Then $\mathrm{Nil}(R) \in \mathrm{Spec}(R)$ by Remark 2.4(a). Let E be an $R_{\mathrm{Nil}(R)}$-module. We show that $A = R\,(+)\,E$ is divided; that is, that each $Q \in \mathrm{Spec}(A)$ is divided in A. By [H, Theorem 25.1(3)], $Q = P\,(+)\,E$ for some $P \in \mathrm{Spec}(R)$. By (3), P is divided in R. It suffices to show that if $a = (r, e) \in A \backslash Q$, then $Q \subseteq Aa$. Consider $q = (s, f) \in Q$. As $s \in P$ and $r \in R \backslash P$, the dividedness of P supplies $t \in R$ such that $s = tr$. Moreover, since $r \in R \backslash P \subseteq R \backslash \mathrm{Nil}(R)$, we have $r^{-1} \in R_{\mathrm{Nil}(R)}$ and so, the $R_{\mathrm{Nil}(R)}$-module structure of E permits us to consider $h := r^{-1}(f - te) \in E$. A calculation reveals that

$$q = (s, f) = (tr, te + rh) = (t, h)(r, e) \in Aa \qquad .$$

We next give "locally divided" analogues of the preceding two results.

Lemma 2.15. *Let R be a ring and E an R-module. Suppose that $A = R\,(+)\,E$ is a locally divided ring. Then*:

(a) *R is a locally divided ring*.

(b) *Suppose that for each $M \in \mathrm{Max}(R)$, $s \in R_M \backslash \mathrm{Nil}(R_M)$, $f \in E_M$, $sf =$* 0 implies $f = 0$. *Then for each $M \in \mathrm{Max}(R)$*, the *R_M-module structure on E_M is induced by a $(\mathrm{Nil}(R_M))^{-1}(R_M)$-module structure on E_M*.

Proof. (a) follows from the second assertion in Proposition 2.1(a). In view of Proposition 2.1(c), the conclusion in (b) is a direct consequence of Lemma 2.13(b) and the following two facts. $\mathrm{Max}(A) = \{M\,(+)\,E : M \in \mathrm{Max}(R)\}$, [H, Theorem 25.1]; and if $Q = M\,(+)\,E \in \mathrm{Max}(A)$, then $A_Q \cong R_M\,(+)\,E_M$ [H, Corollary 25.5(2)]. □

Proposition 2.16. *Let R be a ring. Then the following conditions are equivalent*:

(1) *Each maximal ideal of R contains only one minimal prime ideal of R. Moreover, $R\,(+)\,E$ is a locally divided ring for each R-module E satisfying the following condition: for each $M \in \mathrm{Max}(R)$, the R_M-module structure on E_M is induced by a $(\mathrm{Nil}(R_M))^{-1}(R_M)$-module structure on E_M*;

(2) *Each maximal ideal of R contains only one minimal prime ideal of R. Moreover, $R\,(+)\,E$ is a locally divided ring for some R-module E satisfying the following condition: for each $M \in \mathrm{Max}(R)$, the R_M-module structure on E_M is induced by a $(\mathrm{Nil}(R_M))^{-1}(R_M)$-module structure on E_M*;

(3) *R is a locally divided ring*.

Proof. The condition that each $M \in \mathrm{Max}(R)$ contains a unique element of $\mathrm{Min}(R)$ is included in (1) and (2) to ensure that $\mathrm{Nil}(R_M) \in \mathrm{Spec}(R_M)$. In view of Proposition 2.1(c), (d), the assertion follows from Proposition 2.14 and the facts recalled in the proof of Lemma 2.15. □

The above results simplify over domains, leading to new examples of (locally) divided rings.

Corollary 2.17. *Let R be a domain, with quotient field K. Then*:

(a) *Let E be an R-module. Suppose that $R(+)E$ is a divided (resp., locally divided) ring. Then R is also a divided (resp., locally divided) ring. If, in addition, E is a torsion-free R-module, then for each $M \in \mathrm{Max}(R)$, the R_M-module structure on E_M is induced by a K-vector space structure on E_M.*

(b) *The following conditions are equivalent*:

(1) *$R(+)E$ is a locally divided (resp., divided) ring for each R-module E satisfying the following condition: for each $M \in \mathrm{Max}(R)$, the R_M-module structure on E_M is induced by a K-vector space structure on E_M*;

(2) *$R(+)E$ is a locally divided (resp., divided) ring for some R-module E satisfying the following condition: for each $M \in \mathrm{Max}(R)$, the R_M-module structure on is induced by a K-vector space structure on E_M*;

(3) *$R(+)E$ is a locally divided (resp., divided) ring for each K-vector space E*;

(4) *$R(+)E$ is a locally divided (resp., divided) ring for some K-vector space E*;

(5) *R is a locally divided (resp., divided) domain.*

One of the most important examples of a locally divided domain is a Prüfer domain. A domain R is a Prüfer domain if and only if R is an integrally closed locally divided finite conductor domain ([cf. [M1, Theorem 1]). One may ask if the preceding assertion extends to rings. (Recall [G1] that a ring A is called a *finite conductor ring* if $Aa \cap Ab$ and $\mathrm{Ann}(c)$ are finitely generated ideals of A for all $a, b, c \in A$.) Now, any Prüfer ring is integrally closed [G, Theorem 13]. However, by part (a) of the next example, the rest of the "only if" assertion does not extend to rings. Part (b) gives more non-domain-like behavior of "locally divided".

Example 2.18. (a) There exists a Prüfer ring which is neither locally divided nor a finite conductor ring.

(b) There exists a quasilocal ring R such that $\dim(R) = 1$ and R is not a (locally) divided ring.

Proof. (a) Let K be a field and let $T = \prod_{n\geq 1} K[[X]]$, the product of denumerably many copies of $K[[X]]$. View $K \subseteq T$ via the diagonal map. Consider the ideal $M := \oplus_{n\geq 1} K[[X]]$ of T. We show that $R = K + M$ is as asserted.

(R, M) is quasilocal; and $Z(R) = M$, whence $R = tq(R)$. It follows that R is a Prüfer ring (use the criterion [G, Theorem 13] that each of its overrings is R-flat). To prove that R is not (locally) divided, use the criterion in [B2, Proposition 2], noting that $a = (X, 0, 0, 0, \ldots)$, $b = (0, X, 0, 0, \ldots) \in R$ are such that $a \nmid b$ and $b \nmid a^n$ for $n \geq 1$. Moreover, R is not a finite conductor ring, since $\mathrm{Ann}(a) = 0 \oplus \ \oplus_{n\geq 2} XK[[X]]$ is not finitely generated in R.

(b) Choose R to be any quasilocal domain such that $\dim(R) = 1$. Then $A := R(+)R$ is quasilocal and one-dimensional by [H, Theorem 25.1(3)]. However, Corollary 2.17(a) ensures that A is not (locally) divided, since R is not a vector space over its quotient field. □

Prüfer rings are known to exhibit additional pathology. For instance, Lucas [L, Example 2.11] shows that the localization of a Prüfer ring at a maximal ideal need not be Prüfer. Nevertheless, we next obtain domain-like behavior for the "Prüfer ring" property in the presence of the "$Z(R) = \mathrm{Nil}(R)$" hypothesis. Both parts of Proposition 2.19 fail without this hypothesis.

Proposition 2.19. *Let R be a ring such that $Z(R) = \mathrm{Nil}(R)$. Then*:

(a) *R is a Prüfer ring if and only if R_M is a Prüfer ring for each $M \in \mathrm{Max}(R)$.*

(b) *If R is a Prüfer ring, then R is locally divided.*

Proof. (a) The "if" assertion is valid for arbitrary rings [L, Proposition 2.10]. Conversely, suppose that R is a Prüfer ring. We show that R_Q is a Prüfer ring for each $Q \in \mathrm{Spec}(R)$. Since $Z(R) = \mathrm{Nil}(R)$, we may view $R \subseteq R_Q \subseteq tq(R)$ by Proposition 2.5(a). So, each overring of R_Q is an overring of R. The conclusion follows from the criterion [G, Theorem 13] that a ring is a Prüfer ring if and only if each overring is integrally closed.

(b) By Theorem 2.10, R is locally divided if and only if $R + PR_P$ is R-flat for each $P \in \mathrm{Spec}(R)$. Using Proposition 2.5(a), we may view each $R + PR_P$ as an overring of R. The conclusion follows since each overring of a Prüfer ring is flat [G, Theorem 13]. □

Let R be a Prüfer domain and E a vector space over the quotient field of R. Then $A = R(+)E$ is a Prüfer ring by a result of Lucas [L, Proposition 3.1 (b)] (cf. [H, Theorem 25.11(2)]). Of course, $Z(A) = \mathrm{Nil}(A)$; and A is not an domain if $E \neq 0$. Corollary 2.17 and Proposition 2.19(b) each may be used to show that A is a locally divided. By [H, Theorem 25.1(3)], $\dim(A) = \dim(R)$ can be any preassigned value n, $0 \leq n \leq \infty$.

We focus next on the large and regular quotient rings, $R_{[P]}$ and $R_{[P]}$, respectively, and the variants of the "locally divided" concept that they lead to. They lead to the *same* variant, for Corollary 2.24(c) shows that a ring R is such that $R_{[P]}$ is divided for each $P \in \mathrm{Spec}(R)$ if and only if $R_{[P]}$ is divided for each $P \in \mathrm{Spec}(R)$. Let us say that a ring R satisfies (*) if these equivalent conditions hold. Examples of rings satisfying (*) are given in Example 2.20(b). As those examples suggest, (*) implies "locally divided" (Corollary 2.24(e)), but the converse is false (Example 2.20(a)).

Recall that a ring R is said to *have few zero-divisors* in case $Z(R)$ is expressible as the union of finitely many prime ideals of R. Thus, if $Z(R) \in \mathrm{Spec}(R)$, then R has few zero-divisors. For our purposes, the most important examples of rings R having few zero-divisors are the rings R such that $Z(R) = \mathrm{Nil}(R)$ and the quasilocal treed rings (by Remark 2.4 (a)). (Noetherian rings also have few zero-divisors: Cf. [H, Theorem 7.2].) We often use the result of Griffin [G, Lemma 4] that $R_{[P]} = R_{[P]}$ if R is a ring with few zero-divisors and $P \in \mathrm{Spec}(R)$.

Example 2.20. (a) There exists a Prüfer ring R such that R has few zero-divisors, R is locally divided and for each $P \in \mathrm{Spec}(R)$, neither $R_{[P]}$ nor $R_{[P]}$ is divided.

(b) If R is locally divided and $Z(R) = \mathrm{Nil}(R)$, then $R_{[P]}$ and $R_{[P]}$ are divided for each $P \in \mathrm{Spec}(R)$.

Proof. (a) Let $n \geq 2$ be a positive integer. For $i = 1, \ldots, n$, let R_i be a zero-dimensional ring and put $R = R_i \times \cdots \times R_n$. Since $\dim(R) = \max\{\dim(R_i)\} = 0, R = tq(R)$ (cf. [K, Theorem 84]). Hence, R is a Prüfer ring [G, Theorem 2.13]. Moreover, R is locally divided by Corollary 2.2 (or Proposition 2.1 (b)). As $\mathrm{Spec}(R) = \coprod \mathrm{Spec}(R_i)$ has cardinality $n < \infty$, it follows that R has few zero-divisors. Since $R \subseteq R_{[P]} \subseteq R_{[P]} \subseteq tq(R) = R$ for each $P \in \mathrm{Spec}(R)$, we see that $R_{[P]} = R_{[P]} = R$, which is not divided since it has $n \geq 2$ maximal ideals.

(b) If $P \in \mathrm{Spec}(R)$, then Proposition 2.5(a) yields that $R_{[P]} = R_{(P)} = R_{[P]}$, which is divided. □

Proposition 2.22 pursues the phenomena in Example 2.20. First, we state a lemma used in Theorem 2.23.

Lemma 2.21. *Let R be a ring and T an overring of R. Then*:

(a) $Z(T) \in \mathrm{Spec}(T)$ *if and only if* $Z(R) \in \mathrm{Spec}(R)$.

(b) $\mathrm{Nil}(T) \in \mathrm{Spec}(T)$ *if and only if* $\mathrm{Nil}(R) \in \mathrm{Spec}(R)$.

(c) *If T is a quasilocal treed ring*, *then* $Z(R)$ *and* $\mathrm{Nil}(R)$ *are prime ideals of R.*

Proposition 2.22. *Let R be a locally divided ring. Then the following conditions are equivalent*:

(1) *$R_{[P]}$ is a divided ring for some $P \in \mathrm{Spec}(R)$;*
(2) *$R_{(P)}$ is a divided ring for some $P \in \mathrm{Spec}(R)$;*
(3) *There exists $Q \in \mathrm{Spec}(R)$ such that $Z(R) \subseteq Q$;*
(4) *$Z(R) \in \mathrm{Spec}(R)$.*

Proof. (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (4): Any divided ring is quasilocal and treed. Apply Lemma 2.21(c).

(4) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1), (2): Let Q be as in (3). Then Proposition 2.5(a) gives $R_{[Q]} = R_{(Q)} = R_Q$, which is divided, by the hypothesis on R. Then (1) and (2) follow, with $P = Q$. □

Part (a) of the next result gives a new sufficient condition for a ring to have few zero-divisors.

Theorem 2.23. *Let R be a ring and $P \in \mathrm{Spec}(R)$ such that either $R_{[P]}$ or $R_{(P)}$ is a divided ring. Then:*

(a) *$Z(R)$ and $\mathrm{Nil}(R)$ are each prime ideals of R, and hence R has few zero-divisors.*

(b) *$R_{[Q]} = R_{(Q)}$ for each $Q \in \mathrm{Spec}(R)$.*

(c) *If P is a regular ideal of R, then $Z(R) \subset P$.*

(d) *Let $Q \in \mathrm{Spec}(R)$ such that $Q \subseteq P$. Then R_Q and $R_{[Q]} = R_{(Q)}$ are divided rings.*

Proof. (a) The first assertion follows from Lemma 2.21(c); the final assertion then follows since $Z(R) \in \mathrm{Spec}(R)$.

(b) Combine (a) with [G, Lemma 4].

(c) Deny. Thus, we can choose $x \in Z(R)\backslash P$ and a regular element $y \in P$. By (b) and the hypothesis, $T := R_{[P]}$ is a divided ring. Since $Q = [P]R_{[P]} := \{u \in tq(R) : \text{there exists } s \in R\backslash P \text{ such that } su \in P\}$ is a prime ideal of T, we have that Q is divided in T. However, $x \notin Q$ (since $x \in R\backslash P$ and P is a prime ideal of R), and so $P \subseteq Q \subseteq Tx$. In particular, $y \in Tx \subseteq TZ(R) \subseteq Z(T)$. As $Z(B) \cap R = Z(R)$ for each overring B of $R, y \in Z(R)$, a contradiction.

(d) There are two cases. Suppose first that Q is a regular ideal of R. We claim that $Z(R) \subset Q$. To see this, note first, via (b) and the hypothesis, that $T = R_{[P]}$ is a divided ring. Consider $W = [Q]R_{[P]} = \{u \in tq(R) : \text{there exists } s \in R\backslash P \text{ such that } su \in Q\}$. Observe that $W \in \mathrm{Spec}(T)$ and $Q \subseteq W$. In particular, W is a regular divided prime of T. So, by Proposition 2.3, $Z(T) \subseteq W$. Hence, $Z(R) = Z(T) \cap R \subseteq W \cap R = Q$.

As $Z(R) \subseteq Q \subseteq P$, an application of Proposition 2.5(a) gives $R_P = R_{[P]} = R_{(P)}$ and $R_Q = R_{[Q]} = R_{(Q)}$. As R_P is a divided ring, so is $(QR_P)^{-1}(R_P) \cong R_Q$.

In the remaining case, $Q \subseteq Z(R)$. Then, by (b),

$$R_{[Q]} = R_{(Q)} := R_{(R\setminus Q)\cap(R\setminus Z(R))} = R_{R\setminus z(R)} = tq(R)$$

is divided, since it is a ring of quotients of a divided ring (namely, $R_{[P]}$). It remains to show that R_Q is divided. As in the preceding case, it suffices to prove that R_P is divided. If $P \subseteq Z(R)$, then [Bo, Proposition 7(i), page 65] shows that

$$R_P := R_{R\setminus P} = R_{(R\setminus P)(R\setminus Z(R))} \cong (R_{R\setminus Z(R)})_{im(R\setminus P)}$$

is a ring of fractions of the divided ring $tq(R)$, and hence is divided. Therefore, without loss of generality, P is a regular ideal of R. Then, by (c) and Proposition 2.5(a), $R_P = R_{[P]} = R_{(P)}$, which is divided. □

Parts (c) , (e) of the next result contain assertions regarding property (*) which were promised earlier.

Corollary 2.24. *Let R be a ring. Then*:

(a) *Let* $P \in \operatorname{Spec}(R)$ *such that* $R_{[P]}$ *is a divided ring. Then* $R_{(P)}$ *is either* R_P *or* $tq(R)$.

(b) *Let* $P \in \operatorname{Spec}(R)$ *such that* $R_{(P)}$ *is a divided ring. Then* $R_{(P)}$ *is either* R_P or $tq(R)$.

(c) *The following four conditions are equivalent*:

(1) $R_{[P]}$ *is a divided ring for each* $P \in \operatorname{Spec}(R)$;
(2) $R_{(P)}$ *is a divided ring for each* $P \in \operatorname{Spec}(R)$;
(3) $R_{[M]}$ *is a divided ring for each* $M \in \operatorname{Max}(R)$;
(4) $R_{(M)}$ *is a divided ring for each* $M \in \operatorname{Max}(R)$.

(d) *Let P be a regular prime ideal of R. Then the following three conditions are equivalent*:

(1) $R_{[P]}$ *is a divided ring*;
(2) $R_{(P)}$ *is a divided ring*;
(3) R_P *is a divided ring and* $Z(R) \subset P$.

(e) *If* $R_{[P]}$ (*resp.*, $R_{(P)}$) *is a divided ring for each* $P \in \operatorname{Spec}(R)$, *then R is a locally divided ring*.

Proof. (a), (b): By Theorem 2.23(b), $R_{[P]} = R_{(P)}$. By the proof of Theorem 2.23(d), if P is a regular (resp., nonregular) ideal of R, then $R_{[P]}$ is R_P (resp., $tq(R)$).

(c) Since every prime ideal can be enlarged to a maximal ideal, the assertions follow from Theorem 2.23(b), (d).

(d) $(1) \Leftrightarrow (2)$ by Theorem 2.23(b); $(1) \Rightarrow (3)$ by Theorem 2.23 (d) , (c) and $(3) \Rightarrow (1), (2)$ by Proposition 2.5(a).

(e) Apply Theorem 2.23(d). □

3 GOING-DOWN RINGS

Recall [D1] that a domain R is called a *going-down domain* if $R \subseteq T$ satisfies the going-down property GD for each domain T containing R; and from [D5], that a ring R is called a *going-down ring* if R/P is going-down domain for each $P \in \mathrm{Spec}(R)$. A domain is a going-down ring if only if it is a going-down domain [D5, Remark (a), page 4]; and a ring R is going-down ring if and only if R_M is going-down ring for each $M \in \mathrm{Max}(R)$ [D5, Proposition 2.1 (b)]. Since their introduction in [D2], divided prime ideals and (locally) divided domains have been linked to going-down studies on domains. This section is devoted to developing such connections in the broader context of rings.

Proposition 3.1. *Each locally divided ring is a treed going-down ring.*

Proof. The ''treed'' assertion is immediate from Proposition 2.1(d). In view of the above comments, it remains only to prove that each divided ring is a going-down ring. As appeal to [D5, Remark (c), page 4] (cf. also [B2, Corollary3]) completes the proof. □

Although each going-down domain is treed [D1, Theorem 2.2]), a going-down ring need not be treed [D5, Example 1]. An example of a quasilocal going-down domain which is not a divided domain ([D2, Example 2.9]) shows that the converse of Proposition 3.1 is false. Nevertheless, Theorem 3.4 shows how to use the ''locally divided'' concept to characterize the treed going-down rings within the universe of reduced rings. First, Theorem 3.3 gives a characterization of quasilocal going-down rings within a universe determined by hypotheses that have already proved useful in Section 2.

For the next result, recall the following definition.

A ring extension $A \subseteq B$ is *unibranched* (or: B is a *unibranched extension* of A) if the canonical map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a bijection. Lemma 3.2 generalizes [D2, Lemma 2.3], the corresponding assertion for domains.

Lemma 3.2. *Let $R \subseteq T$ be an integral unibranched ring extension. Then R is a going-down ring if and only if T is a going-down ring.*

Proof. Since integral ring extensions satisfy lying-over [K, Theorem 44], it suffices to prove the following statement. Let $Q \in \mathrm{Spec}(T)$ and $P = Q \cap R \in \mathrm{Spec}(R)$; then $A = R/P$ is going-down domain if and only if $B = T/Q$ is a going-down domain. View $A \subseteq B$ by means of the canonical injective ring-homomorphism $A \to B$. As $A \subseteq B$ inherits integrality from $R \subseteq T$, it suffices by [D2, Lemma 2.3] to show $A \subseteq B$ is unibranched; i.e., by lying-over, that the canonical map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an injection. If

$Q_1 \in \mathrm{Spec}(T)$ and $Q_1 \supseteq Q$, then $(Q_1/Q) \cap A = (Q_1 \cap R)/P$. Since $R \subseteq T$ is unibranched, the conclusion follows from a homomorphism theorem. $\square$

The next result on divided-going-down interplay generalizes a result on domains [D2, Theorem 2.5].

Theorem 3.3. *Let R be a ring such that* $\mathrm{Nil}(R) = Z(R)$ *is divided prime ideal of R. Then the following two conditions are equivalent*:

(1) *R has divided integral unibranched overring*;
(2) *R is a quasilocal going-down ring*.

Moreover, when these conditions hold, R is treed.

Proof. Suppose (1) and (2) hold, with T an overring of R as in (1). Since T is divided, $\mathrm{Spec}(T)$ is linearly ordered by inclusion [B2, Proposition 1(a)]. The same is true for $\mathrm{Spec}(R)$, as integrality ensures that $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is surjective. The "Moreover" assertion follows.

(1) $\Rightarrow$ (2): Assume (1). By integrality, R inherits the quasilocal property from any divided integral overring. As (locally) divided rings are going-down rings by Proposition 3.1, (2) follows via Lemma 3.2 and (1).

(2) $\Rightarrow$ (1): Assume (2). Since $Z(R) = \mathrm{Nil}(R)$, [D5, Proposition 2.3] yields that R has a unique minimal prime ideal, P, and $tq(R) = R_p$. By hypothesis, $P = \mathrm{Nil}(R)$ is a divided ideal of R, and so $PR_p = P$ by Proposition 2.5(c). With $D = R/P$, we have the pullback description

$$R = R + PR_P = R_P \times_{(P^{-1}R)/P} D$$

arising from the surjection $\Pi : R_P \to R_P/PR_P = R_P/P$ and the inclusion $D \to R_P/P = tq(D)$.

Since D inherits from R the property of being a quasilocal going-down ring [D5, Proposition 2.1 (b)], it follows from [D2, Theorem 2.5] that D has a divided integral unibranched overring. Choose one such overring E of D; put $A = \Pi^{-1}(E)$. As A is an overring of R, Lemma 2.6 ensures that $Z(A) = \mathrm{Nil}(A)$. Moreover, $P = PR_P \cap A \in \mathrm{Spec}(A)$ and $A/P \cong E$ is divided. In addition, $A_P = R_P$ is divided by Corollary 2.2, since the minimality of P yields $\dim(R_P) = 0$. So, by Theorem 2.7, A is divided.

It suffices to prove that A is an integral unibranched extension of R. The integrality of $R \subseteq A$ follows from the integrality of $D = R/P \subseteq A/P = E$: this may be seen by applying [F, Corollary 1.5(5)] to the above pullback description of R. Finally, the "unibranched" assertion follows from the "divided integral" assertion. Indeed, since T divided implies $\mathrm{Spec}(T)$ linearly ordered by inclusion [B2, Proposition 1(a)], the "unibranched" conclusion follows by using the lying-over and incomparable properties of integrality [K, Theorem 44]. $\square$

Pursuing the subject of "integral unibranched" ring extensions, we turn to arguably the most important such extensions: $A \subseteq A^+$, where A^+ is the seminormalization (in the sense of [S, page 218]) of a reduced ring A. Theorem 3.4, which completes a thrust begun in [D2], is set in a context far removed from that of Theorem 3.3, for any reduced ring R which satisfies $Z(R) = \mathrm{Nil}(R)$ must be a domain. Note that while much of this project is designed to generalize domain-theoretic studies, Theorem 3.4 is new even if R is a domain see Corollary 3.6.

Theorem 3.4. *Let R be a reduced ring. Then the following conditions are equivalent*:

(1) *R^+ is a locally divided ring*;
(2) *R^+ is a treed going-down ring*;
(3) *R is a treed going-down ring.*

Proof. (1) $\Rightarrow$ (3): The canonical continuous map $\mathrm{Spec}(R^+) \to \mathrm{Spec}(R)$ is Zariski-closed [K, Theorem 44] and bijective, hence a homeomorphism, and hence an order-isomorphism. In particular, since R^+ is treed by Proposition 2.1(d), R is also treed. As the property of being a going-down ring is a local property, it suffices to show that R_M is a going-down ring for each $M \in \mathrm{Max}(R)$.

R_M inherits "reduced" from R, and so R_M has a seminormalization. Since $R_M \subseteq R_M^+$ is integral and unibranched, Lemma 3.2 shows that it is enough to prove that R_M^+ is a going-down ring. Hence, by Proposition 3.1, it suffices to prove that R_M^+ is (locally) divided.

Now, if N is the unique prime ideal of R^+ such that $N \cap R = M$, it follows from (1) that R_N^+ is locally divided. Thus, it suffices to prove $R_M^+ \cong R_N^+$. As [S, Corollary 4.6] ensures that $R_M^+ \cong R_{R\backslash M}^+$, we need only show that $R_{R\backslash M}^+ \cong R_N^+ := (R^+\backslash N)^{-1}(R^+)$. Since $R\backslash M \subseteq R^+\backslash N$, an appeal to [Bo, Proposition 8, page 66] reduces our task to proving that if $Q \in \mathrm{Spec}(R^+)$ and $Q \cap (R^+\backslash N) \neq \phi$, then $Q \cap (R\backslash M) \neq \phi$. Put $P = Q \cap R$. As $Q \nsubseteq N$, the order-isomorphism $\mathrm{Spec}(R^+) \to \mathrm{Spec}(R)$ yields $P \nsubseteq M$. Pick $r \in P\backslash M$. Then $Q\cap$ $(R\backslash M)$ contains r, and hence is nonempty.

(3) $\Longleftrightarrow$ (2): Use the order-isomorphism $\mathrm{Spec}(R^+) \to \mathrm{Spec}(R)$ and Lemma 3.2 to transfer the "treed" and "going-down ring" properties between R and R^+.

(3) $\Rightarrow$ (1): We show that R_N^+ is a divided ring for each $N \in \mathrm{Max}(R^+)$. Put $M = N \cap R$. By integrality, $M \in \mathrm{Max}(R)$; and, by the above argument, $R_N^+ \cong R_M^+$. It suffices to prove that $D = R_M^+$ is divided.

In view of (3) and the order-isomorphism $\mathrm{Spec}(D) \to \mathrm{Spec}(R_M)$, D is quasilocal and treed. Hence, $\mathrm{Nil}(D)$ is the unique minimal prime ideal of D. Moreover, being a seminormalization, D is seminormal (see [S, page 218, lines 1–2]) and hence reduced. Thus, $0 = \mathrm{Nil}(D) \in \mathrm{Spec}(D)$, and so D is a domain. In fact, D is a going-down domain, by applying Lemma 3.2 to $R_M \subseteq D$. It remains only to note that any seminormal quasilocal going-down domain is divided; and this is a consequence of [D2, Corollary 2.6] and the "x^2, x^3" criterion for seminormality of a domain [GH]. □

The above proof shows that any ring R satisfying the hypothesis and conditions in Theorem 3.4 must be locally a going-down domain. However, such R need not be a domain: consider $R = D_1 \times D_2 \times \cdots \times D_n$, where each D_i is a going-down domain and $2 \leq n < \infty$.

Corollary 3.5. *Let R be a seminormal ring. Then the following conditions are equivalent*:

(1) *R is a locally divided ring*;
(2) *R is a treed going-down ring.*

Proof. Since R is seminormal, R is reduced and $R^+ = R$ ([S, page 218, lines 1–2)]. Apply Theorem 3.4. □

Corollary 3.6. *Let R be a domain. Then the following conditions are equivalent*:

(1) *R^+ is a locally divided domain*;
(2) *R^+ is a going-down domain*;
(3) *R is a going-down domain.*

Proof. Any going-down domain is treed [D1, Theorem 2.2]; and any domain is reduced. Apply Theorem 3.4. □

We next move beyond hypotheses like "$Z(R) = \mathrm{Nil}(R)$" and "R is reduced" to study arbitrary going-down rings. Proposition 3.7 generalizes a result on domain [D3, Corollary 2.4] and is motivated by [B2, Proposition 21].

Proposition 3.7. *Let R be a going-down ring. Then each* (*nonzero*) *principal regular prime ideal of R is a maximal ideal of R.*

Proof. Deny. Then there exists $P \in \mathrm{Spec}(R)$ and $M \in \mathrm{Max}(R)$ such that P is a principal ideal, $P \nsubseteq Z(R)$ and $P \subset M$. As $P \nsubseteq Z(R)$, the height of P is nonzero (cf. [H, Theorem 2.1]); hence, we can choose $Q \in \mathrm{Spec}(R)$ such that $Q \subset P$. Observe that $D = R/Q$ is a going-down domain [D5, Proposition 2.1 (b)] in which P/Q is a nonzero principal nonmaximal prime, contradicting [D3, Corollary 2.4]. □

Example 3.9 shows that the "regular" hypothesis cannot be deleted from Proposition 3.7. First, we need to develop a method for constructing going-down rings with zero-divisors. Pullback methods have already been developed in [D5, Proposition 2.2] (which was a chief motivation for Theorem 2.7). Motivated by the examples in Section 2, we turn to idealization and give a going-down-theoretic analogue of Proposition 2.14.

Proposition 3.8. *Let R be a ring. Then the following conditions are equivalent*:

(1) *$R(+)E$ is a going-down ring for each R-module E*;
(2) *$R(+)E$ is a going-down ring for some R-module E*;
(3) *R is a going-down ring*.

Proof. Let E be an R-module and put $A = R(+)E$. It suffices to show that A is a going-down ring if and only if R is a going-down ring; i.e., that A/Q is a going-down domain for each $Q \in \mathrm{Spec}(A)$ if and only if R/P is a going-down domain for each $P \in \mathrm{Spec}(R)$. Recall from [H, Theorem 25.1] that $\mathrm{Spec}(A) = \{P(+)E : P \in \mathrm{Spec}(R)\}$. Consider $Q = P(+)E \in \mathrm{Spec}(A)$, with $P \in \mathrm{Spec}(R)$. It suffices to observe that the surjective map $R \to A/Q$, $r \mapsto (r, 0) + Q$, has kernel P, so that $A/Q \cong R/P$. $\square$

Example 3.9. There exists a going-down ring with a nonzero principal nonmaximal prime ideal.

Proof. Take R to be any going-down domain which is not a field and put $A = R(+)R$. By Proposition 3.8, A is a going-down ring. Let $a = (0, 1) \in A$ and $P = Aa = 0(+)R$. Since $A/P \cong R$, the nonzero principal ideal P of A is a nonmaximal prime ideal of A. $\square$

One motivation for [D2, Theorem 2.5] was to generalize the result of McAdam [M2, Corollary 11] stating that a quasilocal integrally closed domain is a going-down domain if and only if it is a divided domain. It thus is natural to ask if Corollary 3.5 remains valid when its "seminormal" hypothesis is changed to "integrally closed." Example 3.10 answers this negatively.

Example 3.10. For each $n, 1 \leq n \leq \infty$, there exists a quasilocal integrally closed treed going-down ring which is not (locally) divided and which has dimension n.

Proof. Choose an n-dimensional valuation domain R, and put $A = R(+)R/M$. By [H, Theorem 25.1(3)], $\dim(A) = \dim(R) = n$. Since R is a going-down domain (hence, a going-down ring), Proposition 3.8 yields that A is also a going-down ring. As $\mathrm{Spec}(A) = \{P(+)R/M : P \in \mathrm{Spec}(R)\}$, A inherits "quasilocal" and "treed" from R. Moreover, since R is integrally closed and $Z_R(R\backslash M) = M$, it follows from [H, Corollary 25.7] that the integral closure of A is $R(+)(R/M)_{R\backslash M} = R(+)R/M = A$; i.e., A is integrally closed. It remains to show that A is not divided. Observe that $Q = 0(+)R/M \in \mathrm{Spec}(A)$. Q is not divided in A, for if we choose a nonzero element $r \in M$, then $a = (r, 1 + M) \in A \backslash Q$ although $Q \nsubseteq \{(r_1 r, r_1 + r_2 r + M) : r_1, r_2 \in R\} = Aa$. $\square$

Remark 3.11. It follows from the idealization construction that the ring A in Example 3.10 satisfies $Z(A) = 0(+)R/M = \mathrm{Nil}(A)$. Moreover, A satisfies (2) but fails to satisfy (1) in the statement of Theorem 3.3. Hence, one cannot remove from Theorem 3.3 the hypothesis that the nilradical be a divided (prime) ideal. (In fact, for the ring A in Example 3.10, $\mathrm{Nil}(A)$ is the ideal Q which was shown directly to be nondivided in Example 3.10. Thus, Theorem 3.3 leads to another proof that Q is nondivided.) In view of the uses to which the "divided" hypothesis was put in the proof of Theorem 3.3, we would argue for the appropriateness of the pullback-theoretic couching of that proof, of its supporting result Theorem 2.7 and, more generally, of the approach in Section 2 to studying locally divided rings B satisfying $Z(B) = \mathrm{Nil}(B)$. In closing, we ask for an equally appropriate mechanism to illuminate and extend the work in Section 2 on locally divided rings B for which $Z(B) \neq \mathrm{Nil}(B)$.

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